Spectral theory and non-Normal operators

R. R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

March 14, 2017

Contents

1 Non-Normal operators: some remarks

There is no general theory that applies to (infinite dimensional) non-Normal operators. However, in some cases one can say something. Below we present a few examples.

Related stuff: Laplace Transforms and Green's functions #01 in *Point sources and Green's functions*
and Normal modes #02 in *Separation of variables and normal modes.*

---

2 Example: $\mathcal{L} = -\frac{d^2}{dx^2} + \mu(x)$, $a < x < b$, Dirichlet BC, μ complex valued

Here we consider the eigenvalue problem for the operator

$$\mathcal{L} = -\frac{d^2}{dx^2} + \mu(x), \quad a < x < b, \quad \text{with homogeneous Dirichlet BC}, \tag{2.1}$$

where $\mu = \mu(x)$ is some complex valued function.[1] This operator is self-adjoint, with the standard scalar product, only when μ is real valued. In general $\mathcal{L}$ is not even normal, since (at least for μ in C^2)

$$\mathcal{L}^\dagger = -\frac{d^2}{dx^2} + \mu^*(x), \quad \Longrightarrow \quad \mathcal{L}\,\mathcal{L}^\dagger - \mathcal{L}^\dagger\,\mathcal{L} = 4\,i\,\sigma'\,\frac{d}{dx} + 2\,i\,\sigma'', \tag{2.2}$$

where † and * denote adjoints and conjugates, $\sigma = \text{Im}(\mu)$, and the primes denote derivatives with respect to x. On the other hand $\mathcal{L}$ is *symmetric* with respect to the form[2]

$$\boldsymbol{Q(f,\, g) = \int_a^b f(x)\, g(x)\, dx}. \tag{2.3}$$

That is $\boldsymbol{Q(\mathcal{L}\, f,\, g) = Q(f,\, \mathcal{L}\, g)}$

Nevertheless *a spectral theorem, in terms of eigenfunctions and generalized eigenfunctions, applies to this operator,* as we show next. For this purpose we consider the pde initial value problem

$$u_t = u_{xx} - \mu\, u, \;\; a < x < b \;\text{ and }\; t > 0, \;\text{ with }\; u(x, 0) = f(x) \tag{2.4}$$

and homogeneous Dirichlet BC. Then the Laplace Transform $\boldsymbol{U = \int_0^\infty e^{-s\,t} u\, dt}$ satisfies

$$-U_{xx} + \mu\, U + s\, U = \mathcal{L}\, U + s\, U = f, \tag{2.5}$$

[1] Say, μ is piecewise smooth. Note that L^2 is probably enough.
[2] For real valued f and g, Q is the standard scalar product. However, $\mathcal{L}$ is not a real operator.

with $U = 0$ at $x = a,\, b$. We write the *resolvent* $R = (\mathcal{L} + s)^{-1}$ using the Green's function.[3] Namely

$$U(s,\, x) = \int_a^b G(x, y, s)\, f(y)\, dy \quad \text{where} \quad G = \frac{1}{W} \begin{cases} v_1(x,\, s)\, v_2(y,\, s) & \text{for} \quad x \leq y, \\ v_1(y,\, s)\, v_2(x,\, s) & \text{for} \quad x \geq y, \end{cases} \tag{2.6}$$

where

1. $v_1 = v_1(x,\, s)$ satisfies $-v_1'' + (\mu + s)\, v_1 = 0$ with $v_1 = 0$ and $v_1' = 1$ at $x = a$. (2.7)

2. $v_2 = v_2(x,\, s)$ satisfies $-v_2'' + (\mu + s)\, v_2 = 0$ with $v_2 = 0$ and $v_2' = 1$ at $x = b$. (2.8)

3. $W = W(s) = v_1'\, v_2 - v_1\, v_2'$ is the Wronskian.[4] (2.9)
Note: $\boldsymbol{W = 0}$ if and only if $\boldsymbol{v_2 \propto v_1}$ and $\boldsymbol{\lambda = -s}$ is an **eigenvalue** for $\boldsymbol{\mathcal{L}}$. (2.10)

Furthermore (see remark **2.1**)

4. $v_1 = v_1(x,\, s)$ is an entire function of s. For $|s| \gg 1$, $\quad v_1 \sim \dfrac{\sinh(\sqrt{s}\,(x-a))}{\sqrt{s}}$. (2.11)

5. $v_2 = v_2(x,\, s)$ is an entire function of s. For $|s| \gg 1$, $\quad v_2 \sim \dfrac{\sinh(\sqrt{s}\,(x-b))}{\sqrt{s}}$. (2.12)

6. W is an entire function. For $|s| \gg 1$, $\quad W \sim -\dfrac{\sinh(\sqrt{s}\,(b-a))}{\sqrt{s}}$. (2.13)

7. For $|s| \gg 1$,

$$G \sim \frac{-1}{\sqrt{s}\,\sinh(\sqrt{s}(b-a))} \begin{cases} \sinh(\sqrt{s}\,(x-a))\, \sinh(\sqrt{s}\,(y-b)) & \text{for} \quad x \leq y. \\ \sinh(\sqrt{s}\,(y-a))\, \sinh(\sqrt{s}\,(x-b)) & \text{for} \quad x \geq y. \end{cases} \tag{2.14}$$

From this last formula we see that G vanishes as $|s| \to \infty$, at least as fast as $1/\sqrt{s}$.

We now invoke the inverse Laplace Transform, which states that

$$u = \frac{1}{2\,\pi\, i} \int_\Gamma U(s,\, x)\, e^{s\,t}\, ds, \tag{2.15}$$

where Γ is a path in the complex plane of the form $s = a + i\,\mu$, with $-\infty < \mu < \infty$ and $a > 0$ large enough. From the results above we know that:

A. $\boldsymbol{U}$ **has only pole singularities, which correspond to the zeros of** $\boldsymbol{W}$**. Let them be** $\{s_n\}$**.**[5]

B. $\boldsymbol{U}$ **vanishes for** $|s| \to \infty$**.**

Using these results we can "move" the path Γ to the left ($a \to -\infty$). Then, every time the path crosses a pole of U, it picks up a contribution from the pole. In particular, if the pole is simple, the contribution is the residue of $U\, e^{s\,t}$ there. Thus (2.15) yields

$$u = \sum_n u_n(x,\, t), \tag{2.16}$$

where u_n is the contribution from the pole at s_n. However, **(2.16) is, generally, not a mode expansion. Generalized modes with time dependences of the form** $\boldsymbol{t^j\, e^{s_n\, t}}$ **can arise as well.**

Remark 2.1 *The asymptotic estimates in (2.11 – 2.12) follow from WKBJ theory.*
To see that v_1 is an entire function of s, note that it can be written as the solution to the Volterra integral equation[6]

$$v_1(x,\, s) = \frac{\sinh(\sqrt{s}\,(x-a))}{\sqrt{s}} + \int_a^x \frac{\sinh(\sqrt{s}\,(x-y))}{\sqrt{s}}\, \mu(y)\, v_1(y,\, s)\, dy, \tag{2.17}$$

with a similar equation for v_2 — note that $S(\chi,\, s) = (1/\sqrt{s})\, \sinh(\sqrt{s}\,\chi)$ is an entire function of s.

[3] **By definition: the points** $\boldsymbol{\lambda = -s}$ **at which** $\boldsymbol{R}$ **is singular constitute the spectrum of** $\boldsymbol{\mathcal{L}}$**.**
[4] It is easy to see that $W' = 0$, so that W depends on s only.
[5] Note that we know nothing about these zeros, other than there is a countable number of them, and they are isolated. There cannot be a finite number, else W would be a polynomial, which (2.13) excludes.
[6] Differentiating (2.17) it is easy to show that its solution satisfies the problem for v_1.

We can write (at least formally) $v_1 = \sum_0^\infty I_n$ *(2.18)*
where $I_0(x,\, s) = S(x-a,\, s)$ *and* $I_{n+1}(x,\, s) = \int_a^x S(x-y,\, s)\,\mu(y)\,I_n(y,\, s)\,dy$*. However, for* $n > 0$,

$$I_n = \int_{a<y_1<y_2<\cdots<x} S(x-y_n)\,\mu(y_n)\,S(y_n-y_{n-1})\,\mu(y_{n-1})\,\ldots\,S(y_2-y_1)\,\mu(y_1)\,S(y_1-a)\,d\vec{y}, \tag{2.19}$$

where, for simplicity, we have not displayed the argument s*. Since the argument* ξ *for* S *is restricted to the range* $0<\chi<b-a$*, it should be clear that: in any bounded region of the* s*-complex plane, there is a constant* K *such that* $|S| \leq K$*. Similarly, let* M *be a bound on* μ*. Then (2.19) shows that* $I_n \leq K^{n+1}\,M^n\,\frac{(b-a)^n}{n!}$. *(2.20)*
It follows that (2.18) converges uniformly in any bounded region of the complex plane. Thus it solves (2.17), and it is entire (since each if the I_n*'s is).*

THE END.